

# Perfect fractional matchings in $k$ -out hypergraphs\*

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## Abstract

Extending the notion of (random)  $k$ -out graphs, we consider when the  $k$ -out hypergraph is likely to have a perfect fractional matching. In particular, we show that for each  $r$  there is a  $k = k(r)$  such that the  $k$ -out  $r$ -uniform hypergraph on  $n$  vertices has a perfect fractional matching with high probability (i.e., with probability tending to 1 as  $n \rightarrow \infty$ ) and prove an analogous result for  $r$ -uniform  $r$ -partite hypergraphs. This is based on a new notion of hypergraph expansion and the observation that sufficiently expansive hypergraphs admit perfect fractional matchings. As a further application, we give a short proof of a stopping-time result originally due to Krivelevich.

## 1 Introduction

Hypergraphs constitute a far-reaching generalization of graphs and a basic combinatorial construct but are notoriously difficult to work with. A *hypergraph* is a collection  $\mathcal{H}$  of subsets (“*edges*”) of a set  $V$  of “*vertices*.” Such an  $\mathcal{H}$  is  *$r$ -uniform* (or an  *$r$ -graph*) if each edge has cardinality  $r$  (so 2-graphs are graphs). A *perfect matching* in a hypergraph is a collection of edges partitioning the vertex set. For any  $r > 2$ , deciding whether an  $r$ -graph has a perfect matching is an NP-complete problem [17]; so instances of the problem tend to be both interesting and difficult. Of particular interest here has been trying to understand conditions under which a *random* hypergraph is likely to have a perfect matching.

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The most natural model of a random  $r$ -graph is the “Erdős-Rényi” model, in which each  $r$ -set is included in  $\mathcal{H}$  with probability  $p$ , independent of other choices. One is then interested in the “threshold,” roughly, the order of magnitude of  $p = p_r(n)$  required to make a perfect matching likely. Here the graph case was settled by Erdős and Rényi [7, 8], but for  $r > 2$  the problem—which became known as Shamir’s Problem following [6]—remained open until [16]. In each case, the obvious obstruction to containing a perfect matching is existence of an isolated vertex (that is, a vertex contained in no edges), and a natural guess is that this is the *main* obstruction. A literal form of this assertion—the *stopping time* version—says that if we choose random edges *sequentially*, each uniform from those as yet unchosen, then we w.h.p.<sup>1</sup> have a perfect matching as soon as all vertices are covered. This nice behavior does hold for graphs [3], but for hypergraphs remains conjectural (though at least the value it suggests for the threshold is correct).

An interesting point here is that taking  $p$  large enough to avoid isolated vertices produces many more edges than other considerations—e.g., wanting a large *expected* number of perfect matchings—suggest. This has been one motivation for the substantial body of work on models of random graphs in which isolated vertices are automatically avoided, notably random *regular* graphs (e.g., [22]) and the  $k$ -out model. The generalization of the latter to hypergraphs, which we now introduce, will be our main focus here.

**The  $k$ -out model.** For a (“host”) hypergraph  $\mathcal{H}$  on  $V$ ,  $\mathcal{H}(k\text{-out})$  is the random subhypergraph  $\cup_{v \in V} E_v$ , where  $E_v$  is chosen uniformly from the  $k$ -subsets of  $\mathcal{H}_v := \{A \in \mathcal{H} : v \in A\}$  (or—but we won’t see this— $E_v = \mathcal{H}_v$  if  $|\mathcal{H}_v| < k$ ), these choices made independently.

The  $k$ -out model for  $\mathcal{H} = K_{n,n}$  (the complete bipartite graph) was introduced by Walkup [21], who showed that w.h.p.  $K_{n,n}(2\text{-out})$  is Hamiltonian, so in particular contains a perfect matching, and Frieze [12] proved the non-bipartite counterpart of the matching result, showing that  $K_{2n}(2\text{-out})$  has a perfect matching w.h.p. (Hamiltonicity in the latter case turned out to be more challenging; it was studied in [9, 13, 4] and finally resolved by Bohman and Frieze [2], who proved  $K_n(3\text{-out})$  is Hamiltonian w.h.p.). The idea of

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<sup>1</sup>As usual we use *with high probability* (*w.h.p.*) to mean with probability tending to 1 as the relevant parameter—here always  $n$ —tends to infinity.

a general host  $G$  was introduced by Frieze and T. Johansson [11]; see also e.g., Ferber *et al.* [10] for (*inter alia*) a nice connection with  $G_{n,p}$ .

For *hypergraphs* the  $k$ -out model seems not to have been studied previously (random regular hypergraphs *have* been considered, e.g., in [5]). Here the two most important examples would seem to be  $\mathcal{H} = K_n^{(r)}$  (the complete  $r$ -graph on  $n$  vertices) and  $\mathcal{H} = K_{[n]^r}$  (the complete  $r$ -partite  $r$ -graph with  $n$  vertices in each part). It is natural to expect that for each of these there is some  $k = k(r)$  for which  $\mathcal{H}(k\text{-out})$  has a perfect matching w.h.p.. Note that, while almost certainly correct, these are likely to be difficult, as either would imply the aforementioned resolution of Shamir's Problem; still, we would like to regard the following linear relaxations as a small step in this direction. (Relevant definitions are recalled in Section 2.)

**Theorem 1.** *For each  $r$ , there is a  $k$  such that w.h.p.  $K_n^{(r)}(k\text{-out})$  admits a perfect fractional matching and  $w \equiv 1/r$  is the only fractional cover of weight  $n/r$ .*

**Theorem 2.** *For each  $r$ , there is a  $k$  such that w.h.p.  $\mathcal{H} = K_{[n]^r}(k\text{-out})$  admits a perfect fractional matching and each minimum weight fractional cover of  $\mathcal{H}$  is constant on each block of the  $r$ -partition.*

Our upper bounds on the  $k$ 's are quite large (roughly  $r^r$ ), but in fact we don't even know that they must be larger than 2 (though this sounds optimistic), and we make no attempt to optimize. In the more interesting case of (ordinary) perfect matchings, consideration of the expected number of perfect matchings shows that  $k$  does need to be at least exponential in  $r$ .

We will make substantial use of the next observation (or, in the  $r$ -partite case, of the analogous Proposition 6, whose statement we postpone), in which the notion of expansion may be of some interest. Recall that an *independent set* in a hypergraph is a set of vertices containing no edges.

**Proposition 3.** *Suppose  $\mathcal{H}$  is an  $r$ -graph in which, for all disjoint  $X, Y \subseteq V$  with  $X$  independent and*

$$|Y| < (r-1)|X|, \tag{1}$$

*there is some edge meeting  $X$  but not  $Y$ . Then  $\mathcal{H}$  has a perfect fractional matching. If, moreover we replace " $<$ " by " $\leq$ " in (1), then  $w \equiv 1/r$  is the only fractional cover of weight  $n/r$ .*

It's not hard to see that for  $r > 2$  the proof of this can be tweaked to give the stronger conclusion even under the weaker hypothesis. (For  $r = 2$  this is clearly false, e.g., if  $G$  is a matching.)

Related notions of expansion (respectively stronger than and incomparable to ours) appear in [18] and [14]. An additional application of Proposition 3, given in Section 4, is a short alternate proof of the following result of Krivelevich [18].

**Theorem 4.** *Let  $\{\mathcal{H}_t\}_{t \geq 0}$  denote the random  $r$ -graph process on  $V$  in which each step adds an edge chosen uniformly from the current non-edges, let  $T$  denote the first  $t$  for which  $\mathcal{H}_t$  has no isolated vertices. Then  $\mathcal{H}_T$  has a perfect fractional matching w.h.p..*

**Outline.** Section 2 includes definitions and brief linear programming background. Section 3 treats  $K_n^{(r)}$ , proving Proposition 3 and Theorem 1, and the corresponding results for  $K_{[n]^r}$  are proved in Section 4. Finally, Section 5 returns to  $K_n^{(r)}$ , using Proposition 3 to give an alternate proof of Theorem 4.

## 2 Preliminaries

Except where otherwise specified,  $\mathcal{H}$  is an  $r$ -graph on  $V = [n]$ . As usual, we use  $[t]$  for  $\{1, 2, \dots, t\}$  and  $\binom{X}{t}$  for the collection of  $t$ -element subsets of  $X$ . Throughout we use  $\log$  for  $\ln$  and take asymptotics as  $n \rightarrow \infty$  (with other parameters fixed), pretending (following a common abuse) that all large numbers are integers and assuming  $n$  is large enough to support our arguments.

We need to recall a minimal amount of linear programming background (see e.g., [20] for a more serious discussion). For a hypergraph  $\mathcal{H}$ , a *fractional (vertex) cover* is a map  $w : V \rightarrow [0, 1]$  such that  $\sum_{v \in e} w(v) \geq 1$  for all  $e \in \mathcal{H}$ ; the *weight* of a cover  $w$  is  $|w| = \sum_v w(v)$ ; and the *fractional cover number*,  $\tau^*(\mathcal{H})$ , is the largest such weight. Similarly a *fractional matching* of  $\mathcal{H}$  is a  $\varphi : \mathcal{H} \rightarrow [0, 1]$  such that  $\sum_{e \ni v} \varphi(e) \leq 1$  for all  $v \in V$ ; the weight of such a  $\varphi$  is defined as for fractional covers; and the *fractional matching number*,  $\nu^*(\mathcal{H})$ , is the *maximum* weight of a fractional matching.

In this context, LP-duality says that  $\nu^*(\mathcal{H}) = \tau^*(\mathcal{H})$  for any hypergraph. For  $r$ -graphs the common value is trivially at most  $n/r$  (e.g., since  $w \equiv 1/r$  is a fractional cover). A fractional matching in an  $r$ -graph is *perfect* if it achieves this bound; that is, if  $\sum \varphi_e = n/r$  (equivalently  $\sum_{e \ni v} \varphi_e = 1 \ \forall v$ , which would be the definition of perfection in a nonuniform  $\mathcal{H}$ ).

Finally, given  $\mathcal{H}$  we say a nonempty  $X \subseteq V$  is  $\lambda$ -*expansive* if for all  $Y \subseteq V \setminus X$  of size at most  $\lambda|X|$ , there is some edge meeting  $X$  but not  $Y$ .

### 3 Proofs of Proposition of 3 and Theorem 1

*Proof of Proposition 3.* It is enough to show that if  $w$  is a fractional cover with  $t_0 := 1/r - \min_v w(v) > 0$ , then  $|w| \geq n/r$ , with the inequality strict if we assume the stronger version of (1). We give the argument under this stronger assumption; for the weaker, just replace the few strict inequalities below by nonstrict ones. Given  $w$  as above, set, for each  $t > 0$ ,

$$W_t = \{v \in [n] : w(v) \leq \frac{1}{r} - t\}, \quad W^t = \{v \in [n] : w(v) \geq \frac{1}{r} + t\}.$$

Since  $w$  is a fractional cover, each edge meeting  $W_t$  must also meet  $W^{t/(r-1)}$  (or the weight on the edge would be less than 1); so, since  $W_t$  is independent, the hypothesis of Proposition 3 gives  $|W^{t/(r-1)}| > (r-1)|W_t|$  for  $t \in (0, t_0]$  (the  $t$ 's for which  $W_t \neq \emptyset$ ).

For  $s \in \mathbb{R}$ , define  $f(s) = |\{v \in [n] : w(v) \geq s\}|$ . Then

$$\begin{aligned} \int_0^1 f(s) ds &= \int_0^1 \sum_{v \in [n]} \mathbf{1}_{\{w(v) \geq s\}} ds \\ &= \sum_{v \in [n]} \int_0^1 \mathbf{1}_{\{w(v) \geq s\}} ds = \sum_{v \in [n]} w(v) = \tau^*(\mathcal{H}). \end{aligned}$$

We also have  $|W^t| = f(1/r + t)$  and  $|W_t| \geq n - f(1/r - t)$ , implying

$$f(1/r + t/(r-1)) \geq (r-1)(n - f(1/r - t)),$$

with the inequality strict if  $t \in (0, t_0]$ . Thus,

$$\begin{aligned} \tau^*(\mathcal{H}) &= \int_0^1 f(s) ds = \int_0^{1/r} f(s) ds + \int_{1/r}^1 f(s) ds \\ &= \int_0^{1/r} f(1/r - t) dt + \int_0^{(r-1)^2/r} \frac{f(1/r + t/(r-1))}{r-1} dt \\ &\geq \int_0^{1/r} \left[ f(1/r - t) + \frac{f(1/r + t/(r-1))}{r-1} \right] dt \\ &> \int_0^{1/r} \left[ f(1/r - t) + (r-1) \frac{n - f(1/r - t)}{r-1} \right] dt = \frac{n}{r}. \end{aligned}$$

□

We should perhaps note that the converse of Proposition 3 is not true in general (failing, e.g., if  $r > 2$  and  $\mathcal{H}$  is itself a perfect matching). But in the graphic case ( $r = 2$ ) the converse *is* true (and trivial), and the proposition provides an alternate proof of the following characterization, which is [19, Thm. 2.2.4] (and is also contained in [1, Thm. 2.1], e.g.).

**Corollary 5.** *A graph has a perfect fractional matching iff  $|N(I)| \geq |I|$  for all independent  $I$ .*

(where  $N(I)$  is the set of vertices with at least one neighbor in  $I$ ).

*Proof of Theorem 1.* Given  $r$ , let (without trying to optimize)  $k = (2r^2)^r$  and  $c = k^{-1/r} = 1/(2r^2)$ , and let  $\mathcal{H} = K_n^{(r)}(k\text{-out})$ . Theorem 1 (with this  $k$ ) is an immediate consequence of Proposition 3 and the next two routine lemmas. (As usual  $\alpha(\mathcal{H})$  is the size of a largest independent set in  $\mathcal{H}$ .)

**Lemma 3.1.** *W.h.p.  $\alpha(\mathcal{H}) < cn$ .*

**Lemma 3.2.** *W.h.p. every  $X \subseteq V(\mathcal{H})$  with  $|X| \leq cn$  is  $(r-1)$ -expansive.*

*Proof of Lemma 3.1.* The probability that  $S \in \binom{[n]}{s}$  is independent in  $\mathcal{H}$  is

$$\left[1 - \frac{(s-1)_{r-1}}{(n-1)_{r-1}}\right]^{sk} < \exp \left[-sk \left(\frac{s-r}{n}\right)^{r-1}\right].$$

(where  $(a)_b = a(a-1)\cdots(a-b+1)$ ), and summing this over  $S$  of size  $cn$  bounds  $\mathbb{P}(\alpha \geq cn)$  by

$$2^n \exp \left[-cnk(c-r/n)^{r-1}\right] = \exp \left[n(\ln 2 - (1-o(1))kc^r)\right],$$

which tends to 0 as desired.  $\square$

*Proof of Lemma 3.2.* For  $X, Y$  disjoint subsets of  $[n]$ , let  $B(X, Y)$  be the event that  $Y$  meets all edges meeting  $X$ . Then, with  $x = |X|$  and  $y = |Y|$ ,

$$\mathbb{P}(B(X, Y)) \leq \left[1 - \frac{(n-y-1)_{r-1}}{(n-1)_{r-1}}\right]^{kx} \leq \left[1 - \left(\frac{n-y-r}{n}\right)^{r-1}\right]^{kx} \leq \left[\frac{r(y+r)}{n}\right]^{kx},$$

the last inequality following from

$$1 - (1-x)^m \leq mx \tag{2}$$

(valid for  $x \in [0, 1]$  and nonnegative integer  $m$ ). The probability that the conclusion of the lemma fails is thus less than

$$\begin{aligned} \sum \binom{n}{rx} \binom{rx}{x} \left[ \frac{r(y+r)}{n} \right]^{kx} &< \sum \left( \frac{ne}{rx} \right)^{rx} 2^{rx} \left[ \frac{r(y+r)}{n} \right]^{kx} \\ &= \sum \left[ (2e)^r \left( \frac{rx}{n} \right)^{k-r} ((r-1) + r/x)^k \right]^x \\ &< \sum \left[ (4er)^r (r(2r-1)x/n)^{k-r} \right]^x = o(1), \end{aligned}$$

where the sums are over  $1 \leq x \leq cn$ . □

□

## 4 Proof of Theorem 2

As in the proof of Theorem 1 we first show that the conclusions of Theorem 2 are implied (deterministically) by sufficiently good expansion and then show that  $K_{[n]^r}(k\text{-out})$  w.h.p. expands as desired. We take  $V = V_1 \cup \dots \cup V_r$  to be our  $r$ -partition (so  $|V_i| = n \ \forall i$ ) and below always assume  $\mathcal{H} \subseteq K_{[n]^r}$ .

**Proposition 6.** *Suppose  $\varepsilon \in (0, 1/2)$  and  $\lambda > 2r^2$  are fixed and  $\mathcal{H}$  satisfies: for any  $i \in [r]$ ,  $T \subseteq V_i$ ,  $U_j \subseteq V_j$  for  $j \neq i$  and  $U = \cup_{j \neq i} U_j$ , there is an edge meeting  $T$  but not  $U$  provided either*

(i)  $|T| \leq \varepsilon n$  and  $|U_j| \leq \lambda |T| \ \forall j \neq i$ , or

(ii)  $|T| \geq \varepsilon n$  and  $|U_j| \leq (1 - \varepsilon)n \ \forall j \neq i$ .

*Then  $\mathcal{H}$  admits a perfect fractional matching, and every minimum weight fractional cover of  $\mathcal{H}$  is constant on each  $V_i$ .*

*Proof.* Define a *balanced assignment* to be a  $w : V \rightarrow \mathbb{R}$  with  $\sum_{v \in V_i} w(v) = 0$  and  $w(e) \geq 0$  for all  $e \in \mathcal{H}$ .

We claim that (under our hypotheses) the only balanced assignment is the trivial  $w \equiv 0$ . To get Proposition 6 from this, let  $f$  be a minimum weight fractional cover, and let  $w_f(v) = f(v) - \sum_{u \in V_i} f(u)/n$ , for each  $i$  and  $v \in V_i$ . Then  $w_f$  is a balanced assignment:  $\sum_{v \in V_i} w_f(v) = 0$  is obvious and nonnegativity holds since  $f(e) \geq 1$  and, by minimality,  $\sum_{v \in V} f(v) \leq n$ . Thus  $w_f \equiv 0$ , implying  $f$  is as promised.

Suppose then that  $w$  is a balanced assignment. For  $X \subseteq V$  and  $t \geq 0$ , set  $X^t = \{v \in X : w(v) \geq t\}$ ,  $X_t = \{v \in X : w(v) < -t\}$ ,  $X^+ = X^0$  and  $X^- = X_0$ , and define the *value* of  $X$  to be  $\psi(X) = \sum_{v \in X} |w(v)|$ . Let  $S = \{i \in [r] : |V_i^-| \leq \varepsilon n\}$  and  $B = [r] \setminus S$ .

**Lemma 4.1.** *If  $X \subseteq V^-$  and  $|X| \leq \varepsilon n$ , then  $\psi(X) \leq r\psi(V^+)/\lambda$ .*

*Proof.* For any  $t > 0$ , note that every edge meeting  $X_t$  meets  $V^{t/(r-1)}$  since otherwise, we could find an edge of negative weight. So since  $|X_t| \leq |X| \leq \varepsilon n$ , condition (i) implies  $|V^{t/(r-1)}| \geq \lambda|X_t|$ . Thus,

$$\begin{aligned}\psi(V^+) &= \int_0^\infty |V^u| du = \frac{1}{r-1} \int_0^\infty |V^{t/(r-1)}| dt \\ &\geq \frac{\lambda}{r-1} \int_0^\infty |X_t| dt = \frac{\lambda}{r-1} \psi(X). \quad \square\end{aligned}$$

**Lemma 4.2.** *If  $|(V_i)_t| \leq \varepsilon n$ , then  $\max_{j \in S} |V_j^{t/(r-1)}| \geq (1-\varepsilon)n$ .*

*Proof.* Since any edge meeting  $(V_i)_t$  meets  $\cup_{j \neq i} V_j^{t/(r-1)}$  and  $|V_j^+| \leq (1-\varepsilon)n$  for  $j \in B$ , there must (see (ii)) be some  $j \in S$  with  $|V_j^{t/(r-1)}| \geq (1-\varepsilon)n$ .  $\square$

We now claim  $\psi(V_i) \leq 2r^2\psi(V)/\lambda$  for all  $i$ . For  $i \in S$ , we do a little better: Lemma 4.1 gives  $\psi(V_i^-) \leq r\psi(V^+)/\lambda$ , and balance (of  $w$ ) then implies  $\psi(V_i) = 2\psi(V_i^-) \leq r\psi(V)/\lambda$ . For  $i \in B$  write  $W$  for  $V_i$  (just to avoid some double subscripts) and set  $T = \sup\{t : |W_t| \geq \varepsilon n\}$ . Then

$$\psi(W^-) = \psi(W_T) + \psi(W^- \setminus W_T) \leq \psi(W_T) + T|W^- \setminus W_T|.$$

Since  $|W_T| < \varepsilon n$ , Lemma 4.1 gives  $\psi(W_T) \leq r\psi(V^+)/\lambda$ . On the other hand,  $|W_t| \geq \varepsilon n$  for  $t \in [0, T)$ , with Lemma 4.2, implies that there is a  $j \in S$  with  $|V_j^{t/(r-1)}| \geq (1-\varepsilon)n$  for all such  $t$ . Thus

$$\begin{aligned}(1-\varepsilon)T|W^- \setminus W_T| &\leq (1-\varepsilon)nT \leq \int_0^T |V_j^{t/(r-1)}| dt \leq \int_0^\infty |V_j^{t/(r-1)}| dt \\ &= (r-1)\psi(V_j^+) \leq r^2\psi(V^+)/\lambda.\end{aligned}$$

So, combining, we have  $\psi(W) = 2\psi(W^-) \leq 2r^2\psi(V)/\lambda$  (establishing the claim) and

$$\psi(V) = \sum_i \psi(V_i) \leq 2r^3\psi(V)/\lambda.$$

But since  $2r^3 < \lambda$ , this forces  $\psi(V) = 0$  and so  $w \equiv 0$ .  $\square$

*Proof of Theorem 2.* Set  $\lambda = 4r^3$ ,  $\varepsilon = (2r\lambda)^{-1}$  and  $k = 2r\varepsilon^{-r}$  (so  $k$  is a little more than  $r^{4r}$ ). We show that w.h.p.  $\mathcal{H} = K_{[n]^r}(k\text{-out})$  is as in Proposition 6. As earlier, let  $B(X, Y)$  be the event that every edge meeting  $X$  meets  $Y$ .



Suppose first that  $T$  and  $U$  are fixed with  $|U_i| = \lambda|T| \leq \lambda\epsilon n$ . Then

$$\mathbb{P}(B(T, U)) \leq \left[1 - \left(1 - \frac{\lambda|T|}{n}\right)^{r-1}\right]^{k|T|} \leq \left(\frac{r\lambda|T|}{n}\right)^{k|T|}.$$

Summing over choices of  $T$  and  $U$  bounds the probability that  $\mathcal{H}$  violates the assumptions of the proposition for some  $T$  and  $U$  as in (i) by

$$\begin{aligned} r \sum_{t=1}^{\epsilon n} \binom{n}{t} \binom{n}{\lambda t}^{r-1} \left(\frac{r\lambda t}{n}\right)^{kt} &\leq r \sum_{t=1}^{\epsilon n} \left(\frac{en}{t}\right)^t \left(\frac{en}{\lambda t}\right)^{\lambda t(r-1)} \left(\frac{r\lambda t}{n}\right)^{kt} \\ &\leq \sum_{t=1}^{\epsilon n} \left[(r\lambda t/n)^{k-r\lambda} \lambda (er)^{r\lambda}\right]^t = o(1). \end{aligned}$$

Now say  $T$  and  $U$  are fixed with  $|T| = \epsilon n$  and  $|U_i| = (1 - \epsilon)n$ . Then

$$\mathbb{P}(B(T, U)) \leq (1 - \epsilon^{r-1})^{k|T|} \leq \exp[-k|T|\epsilon^{r-1}] \leq \exp[-kn\epsilon^r].$$

So summing over possibilities for  $(T, U)$  bounds the probability of a violation with  $T$  and  $U$  as in (ii) by

$$r2^{nr} \exp[-kn\epsilon^r] \leq \exp[n(r - k\epsilon^r)] = o(1). \quad \square$$

## 5 Proof of Theorem 4

We now turn to our proof of Theorem 4, for which we work with the following standard device for handling the process  $\{\mathcal{H}_t\}$ .

Let  $\xi_S$ ,  $S \in \binom{[n]}{r}$ , be independent random variables, each uniform from  $[0, 1]$ , and for  $\lambda \in [0, 1]$ , let  $G(\lambda)$  be the  $r$ -graph on  $[n]$  with edge set  $\mathcal{E}(\lambda) = \{S : \xi_S \leq \lambda\}$ . Members of  $\mathcal{E}(\lambda)$  will be called  $\lambda$ -edges. Note that with probability one,  $G(0)$  is empty,  $G(1)$  is complete, and the  $\xi_S$ 's are distinct.

Provided the  $\xi_S$ 's are distinct, this defines the discrete process  $\{\mathcal{H}_t\}$  in the natural way, namely by adding edges  $S$  in the order in which their associated  $\xi_S$ 's appear in  $[0, 1]$ . We will work with the following quantities, where  $\gamma = \epsilon \log n$  for some small fixed (positive)  $\epsilon$  and  $g$  is a suitably slow  $\omega(1)$ .

- $\Lambda = \min\{\lambda : G(\lambda) \text{ has no isolated vertices}\};$
- $W_\lambda = \{v \in [n] : d_{G(\lambda)}(v) \leq \gamma\};$
- $\sigma = \frac{\log n - g(n)}{\binom{n-1}{r-1}}$  and  $\beta = \frac{\log n + g(n)}{\binom{n-1}{r-1}};$

- $N = \{v : \exists e \in \mathcal{E}(\beta), v \in e, e \cap W_\sigma \neq \emptyset\}$

(so  $N$  is  $W_\sigma$  together with its  $\mathcal{E}(\beta)$ -neighbors).

**Preview.** With the above framework, our assignment is to show that  $G(\Lambda)$  has a perfect matching w.h.p.. Perhaps the nicest part of this—and the point of coupling the different  $G(\lambda)$ ’s—is that, so long as  $\Lambda \in [\sigma, \beta]$ , which we will show holds w.h.p., the desired assertion on  $G(\Lambda)$  follows *deterministically* from a few properties ((b)-(d)) of Lemma 5.1) involving  $G(\sigma)$ ,  $G(\beta)$  *or both*; so by showing that the latter properties hold w.h.p. we avoid the need for a union bound to cover possibilities for  $\Lambda$ . Production of the fractional matching is then similar to (though somewhat simpler than) what happens in [18]: the relatively few vertices of  $W_\Lambda$  (and some others) are covered by an (ordinary) matching, and the hypergraph induced by what’s left has the expansion needed for Proposition 3.

**Lemma 5.1.** *With the above setup (for fixed  $r$ ) and  $Z = n(\log n)^{-1/r}$ , w.h.p.*

- (a)  $\Lambda \in [\sigma, \beta]$ ;
- (b)  $\alpha(G(\sigma)) < Z$ ;
- (c) *no  $\beta$ -edge meets  $W_\sigma$  more than once and no  $u \notin W_\sigma$  lies in more than one  $\beta$ -edge meeting  $N \setminus \{u\}$ ;*
- (d) *each  $X \subseteq V \setminus W_\sigma$  of size at most  $Z$  is  $r$ -expansive in  $G(\sigma)$ .*

*Proof.* For (a), note that the expected number of isolated vertices in  $G(\lambda)$  is  $h(\lambda) := n(1 - \lambda)^{\binom{n-1}{r-1}}$ . The upper bound (i.e.  $\Lambda < \beta$  w.h.p.) then follows from  $h(\beta) = o(1)$ , and the lower bound is given by Chebyshev’s Inequality (applied to the number of isolated vertices).

For (b), we have

$$\begin{aligned}
\mathbb{P}(\alpha(G(\beta)) \geq Z) &< \binom{n}{Z}(1 - \beta)^{\binom{Z}{r}} < (en/Z)^Z \exp \left[ -\beta \binom{Z}{r} \right] \\
&= \exp \left[ Z \log(en/Z) - (1 - o(1))(n/r) \log n(Z/n)^r \right] \\
&= \exp \left[ Z \log(en/Z) - \Omega(n) \right] = o(1).
\end{aligned}$$

The proofs of (c) and (d) are similarly routine but take a little longer. Aiming for (c), set  $p = \mathbb{P}(\zeta \leq \gamma)$ , where  $\zeta$  is binomial with parameters  $\binom{n-2}{r-1}$

and  $\sigma$ . Since  $\mu := \mathbb{E}\zeta \sim \log n$ , a standard large deviation estimate (e.g., [15, Thm. 2.1]) gives

$$p < \exp[-\mu\varphi(-(\mu - \gamma)/\mu)] < n^{-1+\delta},$$

where  $\varphi(x) = (x+1)\log(x+1) - x$  for  $x \geq -1$  and  $\delta \approx \varepsilon \log(1/\varepsilon)$ .

Failure of the first assertion in (c) implies existence of  $S \in K_n^{(r)}$  and (distinct)  $u, v \in S$  with  $S \in G(\beta)$  and  $u, v \in W_\sigma$ . The probability that this occurs for a given  $S, u, v$  is less than  $\beta p^2$  (the  $p^2$  bounding the probability that each of  $u, v$  lies in at most  $\gamma$  edges not containing the other), so the probability that the assertion fails is less than

$$\binom{n}{r} r^2 \beta p^2 \sim nr(\log n)p^2 = o(1).$$

If the second part of (c) fails, then we must be able to find a  $u \notin W_\sigma$  as well as one of the following configurations, in which  $x, y \in W_\sigma$ ,  $S_i \in G(\beta)$ , and  $a, b \in [n]$  (and vertices and edges within a configuration are distinct):

- (i)  $x, S_1, S_2$  with  $x, u \in S_1 \cap S_2$ ;
- (ii)  $x, y, S_1, S_2$  with  $x, u \in S_1$ ,  $y, u \in S_2$ ;
- (iii)  $x, a, S_1, S_2, S_3$  with  $x, u \in S_1$ ,  $x, a \in S_2$ ,  $u, a \in S_3$ ;
- (iv)  $x, y, a, S_1, S_2, S_3$  with  $x, u \in S_1$ ,  $y, a \in S_2$ ,  $u, a \in S_3$ ;
- (v)  $x, a, S_1, S_2, S_3$  with  $x, a \in S_1$ ,  $u, a \in S_2 \cap S_3$ ;
- (vi)  $x, a, b, S_1, S_2, S_3, S_4$  with  $x, a \in S_1$ ,  $x, b \in S_2$ ,  $u, a \in S_3$ ,  $u, b \in S_4$ ;
- (vii)  $x, y, a, b, S_1, S_2, S_3, S_4$  with  $x, a \in S_1$ ,  $y, b \in S_2$ ,  $u, a \in S_3$ ,  $u, b \in S_4$ ;
- (viii)  $x, a, b, S_1, S_2, S_3$  with  $x, a, b \in S_1$ ,  $u, a \in S_2$ ,  $u, b \in S_3$ .

Thus, with  $M = \binom{n-2}{r-2}$ , summing probabilities for these possibilities bounds the probability of violating the second part of (c) by

$$\begin{aligned} & n^2 p M^2 \beta^2 + n^3 p^2 M^2 \beta^2 + n^3 p M^3 \beta^3 + n^4 p^2 M^3 \beta^3 + n^3 p M^3 \beta^3 \\ & + n^4 p M^4 \beta^4 + n^5 p^2 M^4 \beta^4 + n^4 p M^2 \binom{n-3}{r-3} \beta^3 = o(1). \end{aligned}$$

For (d) it is enough to bound (by  $o(1)$ ) the probability that for some (nonempty)  $X \subseteq V$  of size  $x \leq Z$  and  $Y \subseteq V \setminus X$  of size  $rx$ ,

$$\text{there are at least } \gamma x/r \text{ } \sigma\text{-edges meeting both } X \text{ and } Y. \quad (3)$$

For given  $X, Y$  the expected number of such edges is less than

$$x \cdot rx \binom{n-2}{r-2} \sigma < xr^2 \frac{Z \log n}{n-1} =: bx.$$

(The first inequality is a significant giveaway for small  $x$ , but we have lots of room.) So, again using [15, Thm. 2.1], we find that the probability of (3) is less than

$$\exp[-(\gamma x/r) \log(\gamma/(erb))] < \exp[-\Omega(\gamma x \log \log n)],$$

while the number of possibilities for  $(X, Y)$  is less than

$$\binom{n}{x} \binom{n}{rx} < \exp[(r+1)x \log(n/x)] = \exp[O(x \log n)],$$

and the desired  $o(1)$  bound follows.  $\square$

*Proof of Theorem 4.* By Lemma 5.1 it is enough to show that if (a)-(d) of the lemma hold then  $G(\Lambda)$  has a perfect fractional matching; so we assume we have these conditions and proceed (working in  $G(\Lambda)$ ).

According to (c) (and the definition of  $\Lambda$ ),  $G(\Lambda)$  admits a *matching*,  $M$ , covering  $W_\sigma$  (each edge of which contains exactly one vertex of  $W_\sigma$ ). Let  $W$  be the set of vertices covered by  $M$  (so  $W$  consists of  $W_\sigma$  plus some subset of  $N \setminus W_\sigma$ ), and  $H = G(\Lambda) - W$  (as usual meaning that the edges of  $H$  are the edges of  $G(\Lambda)$  that miss  $W$ ). It is enough to show that  $H$  has a perfect fractional matching, which will follow from Proposition 3 if we show

$$\text{each independent set } X \text{ of } H \text{ is } (r-1)\text{-expansive.} \quad (4)$$

*Proof.* Since such an  $X$  is also independent in  $G(\sigma)$ , (b) gives  $|X| \leq Z$ , and (d) then says  $X$  is  $r$ -expansive in  $G(\sigma)$ , *a fortiori* in  $G(\Lambda)$ . On the other hand, since  $X \cap W_\sigma = \emptyset$ , (c) guarantees that the  $\beta$ -edges (so also the  $\Lambda$ -edges) meeting  $X$  and *not* contained in  $V(H)$  can be covered by some  $U \subseteq W$  of size at most  $|X|$  (namely, (c) says each  $x \in X$  lies in at most one such edge). It follows that the  $\Lambda$ -edges meeting  $X$  that *do* belong to  $H$  cannot be covered by  $(r-1)|X|$  vertices of  $V(H) \setminus X$ .  $\square$

$\square$

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